Economic Theory: Final Exam

Repeated Games and Reputations

Niccolò Lomys*

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Question 1 [15 points]

Let $\delta \in (0,1)$. Consider the δ-discounted infinitely repeated game with perfect monitoring whose stage game G has reward matrix given by

		Player 2	
		H	T
Player 1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

Prove that the infinitely repeated game does not have a pure-strategy Nash equilibrium for any $\delta \in (0,1)$.

Question 2 [30 points]

Let $\delta \in (0,1)$. Consider the δ-discounted infinitely repeated game with perfect monitoring whose stage game G has reward matrix given by

		Player 2	
		L	R
Player 1	U	$2,2$	$x,0$
	D	$0,5$	$1,1$

Consider the strategy profile that induces the outcome path

$$(D,L),(U,R),(D,L),(U,R),\ldots,$$

and that, after any unilateral deviation by Player 1 specifies the outcome path

$$(D,L),(U,R),(D,L),(U,R),\ldots,$$

*Toulouse School of Economics, University of Toulouse Capitole; niccolo.lomys@tse-fr.eu.

and after any unilateral deviation by Player 2, specifies the outcome path

$$(U, R), (D, L), (U, R), (D, L), \ldots.$$

Simultaneous deviations are ignored, i.e. are treated as if neither player had deviated.

(a) [8 points] Formally write and represent with a figure the smallest automaton representation of this strategy profile.

(b) [14 points] Suppose $x = 5$. For what values of δ is this strategy profile a subgame perfect Nash equilibrium of the repeated game?

(c) [8 points] Suppose now $x = 4$. How does this change your answer to part (b)?

Question 3 [30 points]

Let $\delta \in (0, 1)$. Consider the δ-discounted infinitely repeated game with perfect monitoring whose stage game G has reward matrix given by

		Player 2		
		L	M	R
Player 1	U	$1, 1$	$3, 0$	$-2, 0$
	M	$0, 3$	$2, 2$	$-2, 0$
	D	$0, -2$	$0, -2$	$-4, -4$

(a) [15 points] Suppose $\delta \geq 1/2$. Show that the set $\{(1, 1), (2, 2)\}$ is pure-action self-generating.

(b) [15 points] Suppose $\delta = 1/3$. Find a bounded pure-action self-generating set that contains the set $\{(2, 2)\}$.

Solution

(a) Set $W := \{(1, 1), (2, 2)\}$.

(i) The payoff vector $(1, 1)$ corresponds to the payoff from the stage game Nash equilibrium (U, L). Thus, $(1, 1)$ is pure-action decomposable on W for any δ (use a constant γ mapping each strategy profile into $(1, 1)$).

(ii) The action profile (M, M) is enforceable on W for $\delta \geq 1/2$. To see this, consider $\gamma \in W^A$ defined pointwise as $\gamma(M, M) := (2, 2)$ and $\gamma(a) := (1, 1)$ for $a \neq (M, M)$. Clearly, for $i = 1, 2$, we have $2 = (1 - \delta)u_i(M, M) + \gamma_i(M, M)$. Moreover, (M, M) is enforceable on W if

$$\begin{aligned} 2 &\geq \max_i \max_{a_i \in A_i} \left[(1 - \delta)u_i(a_i, a_{-i} = M) + \delta\gamma_i(a_i, a_{-i} = M)\right] \\ &\geq (1 - \delta)(3) + \delta, \end{aligned}$$

which is satisfied for any $\delta \geq 1/2$. Therefore, for any $\delta \geq 1/2$, the action profile (M, M) is enforceable on W and the payoff vector $(2, 2)$ is pure-action decomposable on W.

From the arguments in (i) and (ii) it follows immediately that W is a pure-action self-generating set of payoffs for $\delta \geq 1/2$, as was to be shown. ■

(b) Set $W := \{(2, 2), (-2, 0), (0, -2)\}$.

(i) The action profile (M, M) is enforceable on W for $\delta \geq 1/3$. To see this, consider $\gamma \in W^A$ defined pointwise as $\gamma(M, M) := (2, 2)$ and $\gamma(a) := (-2, 0)$ for $a \neq (M, M)$. Clearly, we have $(2, 2) = (1-\delta)u(M, M)+\gamma(M, M)$. Moreover, (M, M) is enforceable on W if

$$\begin{aligned} 2 &\geq \max_i \max_{a_i \in A_i} \left[(1-\delta)u_i(a_i, a_{-i} = M) + \delta\gamma_i(a_i, a_{-i} = M)\right] \\ &\geq (1-\delta)(3) + \delta(0), \end{aligned}$$

which is satisfied for any $\delta \geq 1/3$. Therefore, for any $\delta \geq 1/3$, the action profile (M, M) is enforceable on W and the payoff vector $(2, 2)$ is pure-action decomposable on W.

(ii) The action profile (U, R) is enforceable on W for $\delta \geq 1/3$. To see this, consider $\gamma \in W^A$ defined pointwise as $\gamma(U, R) = \gamma(M, R) = \gamma(D, R) := (-2, 0)$ and $\gamma(a) := (0, -2)$ otherwise. Clearly, we have $(-2, 0) = (1 - \delta)u(U, R) + \gamma(U, R)$. Moreover, (U, R) is enforceable on W if

$$\begin{aligned} -2 &\geq \max\{(1-\delta)u_1(M, R) + \delta\gamma_1(M, R), (1-\delta)u_1(D, R) + \delta\gamma_1(D, R)\} \\ &= (1-\delta)(-2) + \delta(-2) \end{aligned}$$

which is satisfied for any δ, and

$$\begin{aligned} 0 &\geq \max\{(1-\delta)u_2(U, L) + \delta\gamma_2(U, L), (1-\delta)u_2(U, M) + \delta\gamma_2(U, M)\} \\ &= (1-\delta)(1) + \delta(-2), \end{aligned}$$

which is satisfied for any $\delta \geq 1/3$. Therefore, for any $\delta \geq 1/3$, the action profile (U, R) is enforceable on W and the payoff vector $(-2, 0)$ is pure-action decomposable on W.

(iii) The action profile (D, L) is enforceable on W for $\delta \geq 1/3$. To see this, consider $\gamma \in W^A$ defined pointwise as $\gamma(D, L) = \gamma(D, M) = \gamma(D, R) := (0, -2)$ and $\gamma(a) := (-2, 0)$ otherwise. Clearly, we have $(0, -2) = (1 - \delta)u(D, L) + \gamma(D, L)$. Moreover, (D, L) is enforceable on W if [... analogous argument as in (ii) ...] which is satisfied for any $\delta \geq 1/3$. Therefore, for any $\delta \geq 1/3$, the action profile (D, L) is enforceable on W and the payoff vector $(0, -2)$ is pure-action decomposable on W.

From the arguments in (i)–(iii) it follows that, for $\delta = 1/3$, W is a pure-action self-generating set of payoffs that contains $\{(2, 2)\}$, as was to be shown. ■