ECON 121: Intermediate Microeconomics
Solutions to Problem Set 2

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Problem 1

Consider an economy in which there are two goods, 1 and 2, whose prices are $p_1 > 0$ and $p_2 > 0$, respectively. The two goods can only be consumed in non-negative amounts x_1 and x_2, respectively. A consumer has preferences over $\mathbb{R}^2_+$ which are represented by the utility function

$$u\colon \mathbb{R}^2_+ \to \mathbb{R}, \quad (x_1, x_2) \mapsto u(x_1, x_2) := (x_1 + 2)x_2.$$

The consumer's income is $I > 0$.

(a) Formulate the consumer's utility maximization problem, find the first-order conditions for utility maximization, and find the Marshallian demand functions[1] $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$ for goods 1 and 2, respectively. (Note: Use the Lagrangian method. Assume that the budget constraint holds with equality and that the solution is interior (i.e. $x_1 > 0$ and $x_2 > 0$), thus disregarding the non-negativity constraints on x_1 and x_2.) Check that the second order conditions are satisfied.

Solution

Let prices (p_1, p_2) be fixed. When solving the consumer's utility maximization problem (UMP), we fix income and some level $I > 0$ and we address this question: What is the maximum level of utility the consumer can achieve when facing a given set of prices with income I?

There are three equivalent ways to formulate the consumer's utility maximization problem.[2] (i) In class, you have seen that the problem can be stated as

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \; (x_1 + 2)x_2$$

$$\text{subject to} \quad p_1x_1 + p_2x_2 \le I.$$

(ii) Note that (x_1, x_2) must be an element of $\mathbb{R}^2_+$. Hence, the restrictions on (x_1x_2) are equivalent to the following three inequality constraints: $x_1 \ge 0$, $x_2 \ge 0$ and $p_1x_1 + p_2x_2 \le I$. Therefore, the following is and equivalent, and probably the most transparent statement of the consumer's problem:

$$\max_{(x_1,x_2)\in\mathbb{R}^2} \; (x_1 + 2)x_2$$

$$\text{subject to} \quad x_1 \ge 0$$

[1] The Marshallian demand function is also known as Walrasian or uncompensated demand function.

[2] There are many more, but here we only discuss three of them.

$$x_2 \geq 0$$
$$p_1 x_1 + p_2 x_2 \leq I.$$

This formulation helps you not to lose sight of the non-negativity constraints on x_1 and x_2. The constraints $x_1 \geq 0$ and $x_2 \geq 0$ will not play a major role in this course, but it is better to keep them in mind.[3]

(iii) A more compact notation to formulate the consumer's utility maximization problem is the following. Given prices $p := (p_1, p_2)$ and income I, let

$$B(p, I) := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq I\right\}$$

be the consumer's budget set. The consumer's utility maximization problem is

$$\max_{(x_1,x_2)\in B(p,I)} (x_1 + 2)x_2.$$

You can use this last formulation provided it is clear to you that $(x_1, x_2) \in B(p, I)$ is equivalent to the constraints $x_1 \geq 0$, $x_2 \geq 0$ and $p_1 x_1 + p_2 x_2 \leq I$.

Let's now solve this problem. Assuming that the budget constraint holds with equality and disregarding the non-negativity constraints on x_1 and x_2, we can set up the Lagrangian for the utility maximization problem as[4]

$$L(x_1, x_2, \lambda) = (x_1 + 2)x_2 + \lambda (p_1 x_1 + p_2 x_2 - I),$$

where λ is the Lagrange multiplier.

The first order (necessary) conditions for an interior solution to this problem are:

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0 \Longleftrightarrow x_2 + \lambda p_1 = 0,$$
$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \Longleftrightarrow x_1 + 2 + \lambda p_2 = 0,$$
$$\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0 \Longleftrightarrow p_1 x_1 + p_2 x_2 - I = 0.$$

The optimal choices of x_1 and x_2 solve these three equations simultaneously (also note that there are three unknowns since the value of λ is unknown).

The first two equations can be rewritten in such a way that λ is separated on the right hand side; that is, as

$$\frac{x_2}{p_1} = -\lambda$$

and

$$\frac{x_1 + 2}{p_2} = -\lambda.$$

[3]See the following discussion of non-negativity constraints for this utility maximization problem. Moreover, look at Problem 2, where you are actually required to check for corner solutions.

[4]The budget constraint holds with equality because the utility function is strictly increasing in both arguments (*Quiz:* Why?). However, no argument can be provided to exclude corner solutions to this utility maximization problem. (*Quiz:* Why?) More on this in a while.

Equating the left hand sides of the two previous equations gives

$$\frac{x_2}{p_1} = \frac{x_1 + 2}{p_2}$$

or, equivalently,

$$x_2 = \frac{p_1}{p_2}(x_1 + 2) \tag{1}$$

When dividing by p_1 and p_2, we use our assumption that prices are strictly positive. This is always so in this problem set, and so we omit to notice this detail in the remaining part of the solution sheet.

Replacing (1) into the first order condition for λ (equivalent to the budget constraint holding with equality) we obtain

$$p_1 x_1 + p_2 \frac{p_1}{p_2}(x_1 + 2) = I.$$

Solving the last equation for x_1 returns the candidate[5] Marshallian demand function for good 1:

$$x_1(p_1, p_2, I) = \frac{I - 2p_1}{2p_1}. \tag{2}$$

Replacing (2) for x_1 into (1) gives the candidate Marshallian demand function for good 2:

$$x_2(p_1, p_2, I) = \frac{I + 2p_1}{2p_2}.$$

Note that $x_2(p_1, p_2, I)$ is always strictly positive. However, $x_1(p_1, p_2, I) > 0$ if and only if $p_1 < I/2$. Therefore, the solution is interior only for $p_1 < I/2$. Henceforth, we will assume that this condition is satisfied.

Let's now check whether the second order conditions are satisfied. The bordered Hessian for this problem is

$$H_b := \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x_1} & L_{\lambda x_2} \\ L_{x_1\lambda} & L_{x_1 x_1} & L_{x_1 x_2} \\ L_{x_2\lambda} & L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & 0 & 1 \\ p_2 & 1 & 0 \end{bmatrix}.$$

Note that the bordered Hessian does not depend on x_1 and x_2. This is so because of the functional form of the utility function in this problem, but this is not usually the case. When the bordered Hessian depends on x_1 and x_2, you have to replace them with $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$, respectively, before computing the relevant leading principal minors.[6] Since

$$\det \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x_1} \\ L_{x_1\lambda} & L_{x_1 x_1} \end{bmatrix} = \det \begin{bmatrix} 0 & p_1 \\ p_1 & 0 \end{bmatrix} = -{p_1}^2 < 0$$

[5] Recall that first order conditions are necessary, but not sufficient, for an interior maximum. We haven't yet verified the second order conditions.

[6] The leading principal minor of order p, for $p = 1, 2, 3$, of a 3×3 matrix A is the determinant of the $p \times p$ matrix obtained by considering the first p rows and columns of A. The relevant leading principal minors for second order conditions for this kind of problems are those of order $p = 2$ and $p = 3$.

and

$$\det\begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x_1} & L_{\lambda x_2} \\ L_{x_1\lambda} & L_{x_1x_1} & L_{x_1x_2} \\ L_{x_2\lambda} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \det\begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & 0 & 1 \\ p_2 & 1 & 0 \end{bmatrix}$$
$$= 0\cdot\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - p_1\cdot\det\begin{bmatrix} p_1 & 1 \\ p_2 & 0 \end{bmatrix} + p_2\cdot\det\begin{bmatrix} p_1 & 0 \\ p_2 & 1 \end{bmatrix}$$
$$= 0 - p_1(-p_2) + p_2p_1 = 2p_1p_2 > 0,$$

the relevant leading principal minors of H_b have the required pattern. Therefore, the second order conditions are satisfied and we conclude that $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$ are interior solutions to the constrained utility maximization problem. ■

(b) Now formulate the corresponding expenditure minimization problem and find the compensated demand functions[7] $x_1^c(p_1, p_2, \bar{U})$ and $x_2^c(p_1, p_2, \bar{U})$ for the two goods, where $\bar{U} > 0$ is some arbitrary level of utility. (Note again: Use the Lagrangian method. Assume that the solution is interior and that the other relevant constraint holds with equality. Do not check the second order conditions for this problem.)

Solution

Let prices (p_1, p_2) be fixed. When solving the consumer's expenditure minimization problem (EMP), we fix some arbitrary level of utility $\bar{U}$ and we address this question: What is the minimum level of money expenditure the consumer must make facing a given set of prices to achieve a given level of utility $\bar{U}$? In this construction, we ignore any limitations imposed by the consumer's income and simply ask what the consumer would have to spend to achieve some particular level of utility.[8]

Again, we can formulate the expenditure minimization problem in three equivalent ways. (i) The first statement, which you have seen in class, goes as follows:

$$\min_{(x_1,x_2)\in\mathbb{R}^2_+} \; p_1x_1 + p_2x_2$$
$$\text{subject to} \quad (x_1+2)x_2 \geq \bar{U}.$$

(ii) If you want to be explicit about the non-negativity constraints on x_1 and x_2, you may want to use the following formulation:

$$\begin{aligned} \min_{(x_1,x_2)\in\mathbb{R}^2} \; & p_1x_1 + p_2x_2 \\ \text{subject to} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & (x_1+2)x_2 \geq \bar{U}. \end{aligned}$$

Finally, by defining the set of (x_1, x_2) feasible pairs that give utility at least as high as $\bar{U}$ as

$$G\left(\bar{U}\right) := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1+2)x_2 \geq \bar{U}\right\},$$

[7] The compensated demand function is also known as Hicksian demand function.

[8] When the range of the utility function u is contained in $\mathbb{R}_+$, as it is the case for this problem, we require $\bar{U} > 0$. Otherwise, the problem becomes trivial.

you can state the expenditure minimization problem compactly as

$$\min_{(x_1,x_2)\in G(\bar{U})} \quad p_1x_1 + p_2x_2.$$

Disregarding the non-negativity constraints on x_1 and x_2 and assuming that the other constraint holds with equality[9], we can set up the Lagrangian for the expenditure minimization problem as

$$L(x_1, x_2, \lambda) = p_1x_1 + p_2x_2 + \lambda[(x_1 + 2)x_2 - \bar{U}],$$

where λ is the Lagrange multiplier.

The first order (necessary) conditions for an interior solution to this problem are:

$$\frac{\partial L\,(x_1, x_2, \lambda)}{\partial x_1} = 0 \iff p_1 + \lambda x_2 = 0,$$

$$\frac{\partial L\,(x_1, x_2, \lambda)}{\partial x_2} = 0 \iff p_2 + \lambda(x_1 + 2) = 0,$$

$$\frac{\partial L\,(x_1, x_2, \lambda)}{\partial \lambda} = 0 \iff (x_1 + 2)x_2 - \bar{U} = 0.$$

The optimal choices of x_1 and x_2 solve these three equations simultaneously (also note that there are three unknowns since the value of λ is unknown).

Following similar steps as in part (a), the first two equations yield

$$x_2 = \frac{p_1}{p_2}(x_1 + 2). \tag{3}$$

Replacing (3) into the first order condition for λ and solving for x_1 yields the compensated demand function for good 1:

$$x_1{}^c\left(p_1, p_2, \bar{U}\right) = \left(\frac{p_2\bar{U}}{p_1}\right)^{\frac{1}{2}} - 2. \tag{4}$$

Replacing (4) for x_1 into (3) yields the compensated demand function for good 2:

$$x_2{}^c\left(p_1, p_2, \bar{U}\right) = \left(\frac{p_1\bar{U}}{p_2}\right)^{\frac{1}{2}}.$$

We observe that the non-negativity constraint on x_1 is binding for some values of the exogenous variables p_1, p_2 and $\bar{U}$. Henceforth, we assume $\bar{U} > 4p_1/p_2$, which ensures an interior minimizer. Moreover we omit checking second order conditions for this problem (which are satisfied).[10] ■

[9] It is easy to see that the constraint $(x_1 + 2)x_2 \geq \bar{U}$ has to hold with equality at the optimal choices of x_1 and x_2. Try to understand why this is so and, if interested, feel free to ask us.

[10] Here we would need to check second order conditions for a minimum, which are different that those for a maximum.

(c) Compute the indirect utility function $V(p_1, p_2, I)$ and the expenditure function $E(p_1, p_2, \bar{U})$.

Solution

These are simple calculations. Recall that the the indirect utility function $V(p_1, p_2, I)$ is defined as the maximum value function corresponding to the consumer's utility maximization problem: the most utility the consumer can get at prices (p_1, p_2) with income I. It can be computed by evaluating utility at the Marshallian demand. Thus, we have

$$\begin{aligned} V(p_1, p_2, I) := \max_{(x_1,x_2)\in B(p,I)} u(x_1, x_2) &= u\left(x_1\left(p_1, p_1, I\right), x_2\left(p_1, p_1, I\right)\right) \\ &= \left(x_1\left(p_1, p_1, I\right) + 2\right) x_2\left(p_1, p_1, I\right) \\ &= \left(\frac{I - 2p_1}{2p_1} + 2\right) \frac{I + 2p_1}{2p_2} \\ &= \left(\frac{I - 2p_1 + 4p_1}{2p_1}\right) \frac{I + 2p_1}{2p_2} \\ &= \frac{I + 2p_1}{2p_1} \frac{I + 2p_1}{2p_2} \\ &= \frac{(I + 2p_1)^2}{4p_1 p_2}. \end{aligned} \tag{5}$$

The expenditure function $E\left(p_1, p_2, \bar{U}\right)$ is defined as the minimum value function corresponding to the consumer's expenditure maximization problem: the minimum expenditure required to achieve utility $\bar{U}$ at prices (p_1, p_2). It can be computed by evaluating expenditure at the compensated demand. Hence, we have

$$\begin{aligned} E\left(p_1, p_2, \bar{U}\right) := \min_{(x_1.x_2)\in G(\bar{U})} p_1 x_1 + p_2 x_2 &= p_1 x_1^c\left(p_1, p_2 \bar{U}\right) + p_2 x_2^c\left(p_1, p_2 \bar{U}\right) \\ &= p_1 \left[\left(\frac{p_2 \bar{U}}{p_1}\right)^{\frac{1}{2}} - 2\right] + p_2 \left(\frac{p_1 \bar{U}}{p_2}\right)^{\frac{1}{2}} \\ &= \left(p_1 p_2 \bar{U}\right)^{1/2} - 2p_1 + \left(p_1 p_2 \bar{U}\right)^{1/2} \\ &= 2\left(\left(p_1 p_2 \bar{U}\right)^{1/2} - p_1\right). \quad \blacksquare \end{aligned}$$

(d) Is $\frac{\partial x_1^c(p_1,p_2,\bar{U})}{\partial p_1}$ positive or negative? Moreover, verify that

$$-\frac{\frac{\partial V(p_1,p_2,I)}{\partial p_1}}{\frac{\partial V(p_1,p_2,I)}{\partial I}} = x_1\left(p_1, p_2, I\right).$$

This is called Roy's identity.

Finally, verify that

$$\frac{\partial E\left(p_1, p_2, \bar{U}\right)}{\partial p_1} = x_1^c\left(p_1, p_2, \bar{U}\right).$$

This is called Shephard's lemma.

Solution

From part (b) we know that

$$x_1{}^c\left(p_1,p_2,\bar{U}\right)=\left(\frac{p_2\bar{U}}{p_1}\right)^{\frac{1}{2}}-2=\left(p_2\bar{U}\right)^{\frac{1}{2}}p_1^{-\frac{1}{2}}-2.$$

Therefore,

$$\frac{\partial x_1^c\left(p_1,p_2,\bar{U}\right)}{\partial p_1}=-\frac{1}{2}\left(p_2\bar{U}\right)^{\frac{1}{2}}p_1^{-\frac{3}{2}}=-\frac{1}{2}\left(\frac{p_2\bar{U}}{p_1^3}\right)^{\frac{1}{2}},$$

which is negative because $p_1, p_2, \bar{U} > 0$.

From part (c) we know that

$$V(p_1,p_2,I)=\frac{(I+2p_1)^2}{4p_1p_2}.$$

Thus,

$$\frac{\partial V\left(p_1,p_2,I\right)}{\partial p_1}=\frac{16(I+2p_1)p_1p_2-4(I+2p_1)^2p_2}{(4p_1p_2)^2}=\frac{4p_2(I+2p_1)(2p_1-I)}{(4p_1p_2)^2}$$

and

$$\frac{\partial V\left(p_1,p_2,I\right)}{\partial I}=\frac{2(I+2p_1)}{4p_1p_2}.$$

It follows that

$$-\frac{\frac{\partial V(p_1,p_2,I)}{\partial p_1}}{\frac{\partial V(p_1,p_2,I)}{\partial I}}=\frac{4p_2(I+2p_1)(I-2p_1)}{(4p_1p_2)^2}\frac{4p_1p_2}{2(I+2p_1)}=\frac{I-2p_1}{p_1}=x_1\left(p_1,p_2,I\right),$$

where the last equality follows from our derivation of the Marshallian demand function for good 1 in part (a). This establishes Roy's identity for good 1 in our example, as desired.

From part (c) we also know that

$$E\left(p_1,p_2,\bar{U}\right)=2\left(\left(p_1p_2\bar{U}\right)^{1/2}-p_1\right)=2\left(p_1p_2\bar{U}\right)^{1/2}-2p_1.$$

Therefore,

$$\frac{\partial E\left(p_1,p_2,\bar{U}\right)}{\partial p_1}=\frac{1}{2}2\left(p_2\bar{U}\right)^{\frac{1}{2}}p_1^{-\frac{1}{2}}-2=\left(\frac{p_2\bar{U}}{p_1}\right)^{\frac{1}{2}}-2=x_1{}^c\left(p_1,p_2,\bar{U}\right),$$

where the last equality follows from our derivation of the compensated demand function for good 1 in part (b). This establishes Shephard's lemma for good 1 in our example, as desired. ■

(e) Verify that

$$V\left(p_1,p_2,E(p_1,p_2,\bar{U})\right)=\bar{U}\quad\text{and that}\quad E\left(p_1,p_2,V(p_1,p_2,I)\right)=I.$$

Moreover, verify that

$$x_i\left(p_1, p_2, I\right) = x_i^c\left(p_1, p_2, V(p_1, p_2, I)\right) \quad \text{and that} \quad x_i^c\left(p_1, p_2, \bar{U}\right) = x_i\left(p_1, p_2, E(p_1, p_2, \bar{U})\right)$$

for $i = 1, 2$. Explain why these are equal.

Solution

Recall again that in part (c) we found

$$V(p_1, p_2, I) = \frac{(I + 2p_1)^2}{4p_1 p_2} \qquad \text{and} \qquad E\left(p_1, p_2, \bar{U}\right) = 2\left(\left(p_1 p_2 \bar{U}\right)^{1/2} - p_1\right).$$

Therefore,

$$\begin{aligned}
V\left(p_1, p_2, E\left(p_1, p_2, \bar{U}\right)\right) &= \frac{\left(E\left(p_1, p_2, \bar{U}\right) + 2p_1\right)^2}{4p_1 p_2} \\
&= \frac{\left(2\left(p_1 p_2 \bar{U}\right)^{1/2} - 2p_1 + 2p_1\right)^2}{4p_1 p_2} \\
&\frac{\left(2\left(p_1 p_2 \bar{U}\right)^{1/2}\right)^2}{4p_1 p_2} \\
&= \frac{4p_1 p_2 \bar{U}}{4p_1 p_2} \\
&= \bar{U},
\end{aligned}$$

and

$$\begin{aligned}
E\left(p_1, p_2, V(p_1, p_2, I)\right) &= 2\left[\left(p_1 p_2 V(p_1, p_2, I)\right)^{1/2} - p_1\right] \\
&= 2\left[\left(p_1 p_2 \frac{(I + 2p_1)^2}{4p_1 p_2}\right)^{1/2} - p_1\right] \\
&= 2\left[\left(\frac{(I + 2p_1)^2}{4}\right)^{1/2} - p_1\right] \\
&= 2\left(\frac{I + 2p_1}{2} - p_1\right) \\
&= I + 2p_1 - 2p_2 \\
&= I,
\end{aligned}$$

as desired.

This result is quite simple and intuitive. It says that if $E(p_1, p_2, \bar{U})$ is the amount of income required to achieve utility $\bar{U}$, then the most utility a consumer can get with wealth $E(p_1, p_2, \bar{U})$ is exactly $\bar{U}$. Similarly, if $V(p_1, p_2, I)$ is the most utility that a consumer can achieve with wealth I at prices (p_1, p_2), then to achieve utility $V(p_1, p_2, I)$ will take wealth at least I.

Now let's show the equalities for demand functions. From part (a) we know that

$$x_1(p_1, p_2, I) = \frac{I - 2p_1}{2p_1} \quad \text{and} \quad x_2(p_1, p_2, I) = \frac{I + 2p_1}{2p_2}$$

and from part (b) we know that

$$x_1{}^c\left(p_1, p_2, \bar{U}\right) = \left(\frac{p_2\bar{U}}{p_1}\right)^{\frac{1}{2}} - 2 \quad \text{and} \quad x_2{}^c\left(p_1, p_2, \bar{U}\right) = \left(\frac{p_1\bar{U}}{p_2}\right)^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
x_1^c(p_1, p_2, V(p_1, p_2, I)) &= \left(\frac{p_2 V(p_1, p_2, I)}{p_1}\right)^{\frac{1}{2}} - 2 \\
&= \left(\frac{p_2 \frac{(I+2p_1)^2}{4p_1p_2}}{p_1}\right)^{\frac{1}{2}} - 2 \\
&= \left(\frac{(I + 2p_1)^2}{4p_1^2}\right)^{\frac{1}{2}} - 2 \\
&= \frac{I + 2p_1}{2p_1} - 2 \\
&= \frac{I - 2p_1}{2p_1} \\
&= x_1(p_1, p_2, I),
\end{aligned}$$

and

$$\begin{aligned}
x_2^c(p_1, p_2, V(p_1, p_2, I)) &= \left(\frac{p_1 V(p_1, p_2, I)}{p_2}\right)^{\frac{1}{2}} \\
&= \left(\frac{p_1 \frac{(I+2p_1)^2}{4p_1p_2}}{p_2}\right)^{\frac{1}{2}} \\
&= \left(\frac{(I + 2p_1)^2}{4p_2^2}\right)^{\frac{1}{2}} \\
&= \frac{I + 2p_1}{2p_2} \\
&= x_2(p_1, p_2, I),
\end{aligned}$$

as desired.

Finally,

$$\begin{aligned}
x_1\left(p_1, p_2, E(p_1, p_2, \bar{U})\right) &= \frac{E(p_1, p_2, \bar{U}) - 2p_1}{2p_1} \\
&= \frac{2\left(p_1p_2\bar{U}\right)^{1/2} - 2p_1 - 2p_1}{2p_1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(p_1 p_2 \bar{U}\right)^{1/2} - 4p_1}{p_1} \\
&= \frac{\left(p_1 p_2 \bar{U}\right)^{1/2}}{p_1} - 2 \\
&= \left(\frac{p_2 \bar{U}}{p_1}\right)^{\frac{1}{2}} - 2 \\
&= x_1^c \left(p_1, p_2, \bar{U}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
x_2\left(p_1, p_2, E(p_1, p_2, \bar{U})\right) &= \frac{E(p_1, p_2, \bar{U}) + 2p_1}{2p_2} \\
&= \frac{2\left(p_1 p_2 \bar{U}\right)^{1/2} - 2p_1 + 2p_1}{2p_2} \\
&= \frac{\left(p_1 p_2 \bar{U}\right)^{1/2}}{p_2} \\
&= \frac{\left(p_1 p_2 \bar{U}\right)^{1/2}}{p_2} \\
&= \left(\frac{p_1 \bar{U}}{p_2}\right)^{\frac{1}{2}} \\
&= x_2^c \left(p_1, p_2, \bar{U}\right),
\end{aligned}
$$

as desired.

The compensated demand function keeps the consumer's utility level fixed as expenditure (required income) changes, in contrast to the Marshallian demand function which keeps the income fixed but allows utility to vary. The first two relations say that the Marshallian demand at prices (p_1, p_2) and income I is equal to the compensated demand at prices (p_1, p_2) and the utility level that is the maximum that can be achieved at prices (p_1, p_2) and income I. The second two relations say that the compensated demand at any prices (p_1, p_2) and utility level $\bar{U}$ is the same as the Marshallian demand at those prices and an income level equal to the minimum expenditure necessary at those prices to achieve that utility level. ■

(f) Verify that the Slutsky equation for good 1 with respect to its own price holds in this case. To do this, first write the the Slutsky equation for a general utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$, $(x_1, x_2) \mapsto u(x_1, x_2)$. (Note: Assume that the associated Marshallian demand function is differentiable with respect to prices and income and that the compensated demand function is differentiable with respect to prices.) Then use your derivations in (a), (b) and (c) to substitute for the various terms in this expression and show that this equality holds for the current utility function.

Solution

Fix prices $p_1, p_2 > 0$ and income $I > 0$. Consider a general a general utility function. In class, we have seen that the Slutsky equation for good 1 with respect to its own price is of the form

$$\frac{\partial x_1(p_1,p_2,I)}{\partial p_1} = \left.\frac{\partial x_1{}^c\left(p_1,p_2,\bar{U}\right)}{\partial p_1}\right|_{\bar{U}=V(p_1,p_2,I)} - x_1(p_1,p_2,I)\frac{\partial x_1(p_1,p_2,I)}{\partial I}. \tag{6}$$

The first term on the right hand side of the previous equation is the substitution effect, while the second one is the income effect. Recall that the Slutsky equation holds when utility is fixed at a level $\bar{U}$ given by maximizing utility at the original prices and income; that is, at $\bar{U} = V(p_1,p_2,I)$. This is way we evaluate the partial derivative of the compensated demand function at that point.[11] Observe that the assumption that the Marshallian and compensated demand functions are differentiable allows us to take the partial derivatives in the previous expression.

Now consider the utility function whose functional form is defined by $u(x_1,x_2) := (x_1+2)x_2$. Take the partial derivative with respect to p_1 of the associated Marshallian demand for good 1 we derived in part (a) (see (2)). This way we obtain an expression for left hand side of the Slutsky equation (6):

$$\frac{\partial x_1(p_1,p_2,I)}{\partial p_1} = -\frac{I}{2p_1{}^2}.$$

Then, take the partial derivative with respect to p_1 of the compensated demand function for good 1 we derived in (b) (see (4)). This way we obtain an expression for the first term of the right hand side of the Slutsky equation (6):

$$\frac{\partial x_1{}^c\left(p_1,p_2,\bar{U}\right)}{\partial p_1} = -\frac{1}{2}\frac{(p_2\bar{U})^{\frac{1}{2}}}{p_1{}^{\frac{3}{2}}}.$$

Then, we replace $\bar{U}$ in the previous equation with the indirect utility function $V(p_1,p_2,I)$ we derived in part (c) (see (5)) to obtain

$$\left.\frac{\partial x_1{}^c\left(p_1,p_2,\bar{U}\right)}{\partial p_1}\right|_{\bar{U}=V(p_1,p_2,I)} = -\frac{1}{4}\frac{I+2p_1}{p_1{}^2} = -\frac{I+2p_1}{4p_1{}^2}.$$

Finally we multiply the Marshallian demand for good 1 in (2) by its partial derivative with respect to income to obtain an expression for second term of the right hand side of the Slutsky equation (6)

$$x_1(p_1,p_2,I)\frac{\partial x_1(p_1,p_2,I)}{\partial I} = \left(\frac{I-2p_1}{2p_1}\right)\frac{1}{2p_1} = \frac{I-2p_1}{4p_1{}^2}.$$

Combing these expressions we have

$$\begin{aligned}\left.\frac{\partial x_1{}^c\left(p_1,p_2,\bar{U}\right)}{\partial p_1}\right|_{\bar{U}=V(p_1,p_2,I)} - x_1(p_1,p_2,I)\frac{\partial x_1(p_1,p_2,I)}{\partial I} &= -\frac{I+2p_1}{4p_1{}^2} - \frac{I-2p_1}{4p_1{}^2}\\ &= -\frac{I}{2p_1{}^2}\\ &= \frac{\partial x_1(p_1,p_2,I)}{\partial p_1},\end{aligned}$$

which verifies the Slutsky equation for good 1 with respect to its own price holds.

[11] This is the only term in which $\bar{U}$ appears.

For the next two questions of this problem, consider a general utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$, $(x_1, x_2) \mapsto u(x_1, x_2)$ satisfying the usual assumptions economists make. Assume that the associated Marshallian demand function is differentiable with respect to prices and income and that the compensated demand function differentiable is differentiable with respect to prices.[12] ■

(g) The motivation that is commonly given for studying income and substitution effects is that it helps understand the possibility of a Giffen good. Using the Slutsky equation, explain the connection between Giffen goods and inferior goods. In particular, which of the following two statements is necessarily true and which one is not always so? Why?

   (i) A Giffen good is an inferior good;

   (ii) An inferior good is a Giffen good.

Solution

Note again that the Slutsky equation for good i with respect to its own price takes the form

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial {x_i}^c}{\partial p_i} - x_i \frac{\partial x_i}{\partial I}, \tag{7}$$

where we economize on notation because it is now clear what we are talking about (see solution to part (f)).

Recall also that good i is called a Giffen good if $\frac{\partial x_i}{\partial p_i} > 0$, and that good i is said to be an inferior good if $\frac{\partial x_i}{\partial I} < 0$.

Suppose that good i is a Giffen good; that is, $\frac{\partial x_i}{\partial p_i} > 0$. Since $\frac{\partial {x_i}^c}{\partial p_i}$ is negative[13], the Slutsky equation in (7) is satisfied only if $x_i \frac{\partial x_i}{\partial I}$ is negative, i.e. only if $\frac{\partial x_i}{\partial I} < 0$. That is, it must be that good i is inferior. This shows that the first statement is true: any Giffen good is an inferior good.

Now suppose that good i is an inferior good; that is, $\frac{\partial x_i}{\partial I} < 0$, so that the income effect $x_i \frac{\partial x_i}{\partial I}$ is negative. From the Slutsky equation we see that, for the good to be Giffen (i.e., for $\frac{\partial x_i}{\partial p_i} > 0$), we require not only $\frac{\partial x_i}{\partial I}$ to be negative, but also to be large enough in magnitude to outweigh $\frac{\partial {x_i}^c}{\partial p_i}$, the substitution effect, which we know to be always negative. In other words, for the good to be Giffen, we need the income effect to be negative (good i has to be inferior) *and* large enough to outweigh the substitution effect. This shows that an inferior good is not necessarily a Giffen good, which implies that the second statement is false. In particular, the Slutsky equation shows that if the income effect does not dominate the substitution effect, then, even if the good is inferior, it will not be Giffen. ■

(h) Recall that the compensated and Marshallian demand functions for good i, $i = 1, 2$, are related through the equality

$$x_i^c\left(p_1, p_2, \bar{U}\right) = x_i\left(p_1, p_2, E(p_1, p_2, \bar{U})\right),$$

[12] For the mathematically inclined reader: This is so when the utility function is twice continuously differentiable on the interior of $\mathbb{R}^2_+$, strictly increasing and quasi-concave.

[13] We have shown in class that, "under the usual assumptions on utility economists make, compensated demand curves slope downward": when the price of some good increases, the compensated demand for that good decreases. This is an immediate consequence of Shephard's lemma. A simple way to see this graphically is to note that the change in compensated demand given a change in price is a shift along an indifference curve.

that is the point of departure for deriving the Slutsky equation. Now differentiated both sides of the previous equality for good 1 (i.e., $i = 1$) with respect to p_2. What can you say about income effects and whether goods 1 and 2 are substitutes? (*Hint:* Use Shephard's lemma and the fact that $\partial x_1/\partial E = \partial x_1/\partial I$.)

Solution

The compensated and Marshallian demand functions for good 1 are related through the equality

$$x_1^c\left(p_1, p_2, \bar{U}\right) = x_1\left(p_1, p_2, E(p_1, p_2, \bar{U})\right). \tag{8}$$

Differentiating both sides of equation (8) with respect to p_2 we get

$$\frac{\partial x_1{}^c}{\partial p_2} = \frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial E}\frac{\partial E}{\partial p_2},$$

which is equivalent to

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_1{}^c}{\partial p_2} - \frac{\partial x_1}{\partial E}\frac{\partial E}{\partial p_2}. \tag{9}$$

By Shepard's lemma, we know that $\partial E/\partial p_2 = x_2^c$. Moreover, observe that we can write $\partial x_1/\partial E$ as $\partial x_1/\partial I$ (this is just notation). Using these facts, (9) is equivalent to

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_1{}^c}{\partial p_2} - x_2{}^c\frac{\partial x_1}{\partial I}. \tag{10}$$

Since the compensated and Marshallian demand functions for good 2 are related through the equality

$$x_2^c\left(p_1, p_2, \bar{U}\right) = x_2\left(p_1, p_2, E(p_1, p_2, \bar{U})\right),$$

it follows that (10) is equivalent to

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_1{}^c}{\partial p_2} - x_2\frac{\partial x_1}{\partial I}.$$

The previous expression is the Slutsky equation for good 1 with respect to the price of good 2. Now recall that goods 1 and 2 are said to be substitutes if $\frac{\partial x_1}{\partial p_2} > 0$. We know that $\frac{\partial x_1{}^c}{\partial p_2}$ is positive. Hence, if good 1 is inferior (that is, if $\frac{\partial x_1}{\partial I} < 0$), then the goods are necessarily substitutes. If good 1 is normal (that is, if $\frac{\partial x_1}{\partial I} > 0$), then the goods are substitutes when the substitution effect dominates the income effect, and they are complements when the income effect dominates the substitution effect. ■

Problem 2

In most of the utility maximization problems we encounter in this course, the solution is interior. This means that a strictly positive quantity of each good is demanded at the utility maximizing bundle $(x_1^*, \ldots, x_n^*)$ (i.e. $x_i^* > 0$ for $i = 1, \ldots, n$.) However, this is not always the case, as the following problem shows.

Consider again an economy in which there are two goods, 1 and 2, whose prices are $p_1 > 0$ and $p_2 > 0$, respectively. The two goods can only be consumed in non-negative amounts x_1 and x_2, respectively. A consumer has preferences over $\mathbb{R}^2_+$ which are represented by the utility function

$$u \colon \mathbb{R}^2_+ \to \mathbb{R}, \quad (x_1, x_2) \mapsto u(x_1, x_2) := \sqrt{x_1} + x_2.$$

The consumer's income is $I > 0$.

(a) Formulate the consumer's utility maximization problem, writing down explicitly all the constraints (i.e., the budget constraint and the non-negativity constraints on x_1 and x_2).

Solution

The consumer's utility maximization problem formulated in the required form is

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \quad & \sqrt{x_1} + x_2 \\ \text{subject to} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & p_1 x_1 + p_2 x_2 \leq I. \quad \blacksquare \end{aligned}$$

(b) Show that the consumer's utility function is strictly increasing is x_1 and x_2 (hence, the consumer's preferences are monotone).

Solution

For this utility function we have

$$MU_1(x_1, x_2) := \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} = \frac{1}{2\sqrt{x_1}} > 0$$

for all $(x_1, x_2) \in \operatorname{Int}\left(\mathbb{R}^2_+\right)$ and, similarly,

$$MU_2(x_1, x_2) := \frac{\partial u(x_1, x_2)}{\partial x_2} = 1 > 0.$$

Since both marginal utilities are strictly positive in the interior of $\mathbb{R}^2_+$, it immediately follows that the consumer's utility function in strictly increasing in x_1 and x_2. Hence, the consumer's preferences it represents are monotone. $\blacksquare$

(c) Now that you have shown that the utility function is strictly increasing, you know that the budget constraint holds with equality at the utility maximizing bundle. Ignore for a moment the the non-negativity constraints on x_1 and x_2 and solve the utility maximization problem using the substitution method. That is: (a) using the budget constraint, express one choice variable as a function of the other choice variables and the parameters (exogenous variables) p_1, p_2 and I of the model; (b) plug your expression into the objective function; (c) take the first order condition and solve for $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$. Skip the check of second order conditions.

Solution

Since the budget constraint holds with equality, we can rearrange it and express x_2 as a function of x_1 and the exogenous variables as

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2}x_1.$$

Putting this into the objective function gives

$$\max_{x_1 \in \mathbb{R}} \sqrt{x_1} + \frac{I}{p_2} - \frac{p_1}{p_2}x_1.$$

Taking the first order condition and solving it for x_1, we find $x_1(p_1, p_2.I)$:

$$\begin{aligned} \frac{1}{2}\frac{1}{\sqrt{x_1}} - \frac{p_1}{p_2} = 0 \iff \frac{1}{\sqrt{2}} &= 2\frac{p_1}{p_2} \\ \iff \sqrt{x_1} &= \frac{p_2}{2p_1} \\ \iff x_1(p_1, p_2.I) &= \left(\frac{p_2}{2p_1}\right)^2. \end{aligned}$$

Putting $x_1(p_1, p_2.I)$ back for x_1 into the above expression for x_2 yields $x_2(p_1, p_2.I)$:

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2}\left(\frac{p_2}{2p_1}\right)^2 \iff x_2(p_1, p_2.I) = \frac{I}{p_2} - \frac{p_2}{4p_1}. \quad \blacksquare$$

(d) Now it's time to reconsider our non-negativity constraints. Show that $x_1(p_1, p_2, I)$ is always strictly positive. Provide a condition on p_2 under which $x_2(p_1, p_2, I)$ is strictly positive.

Solution

Since prices are strictly positive and the square of a strictly positive real number is strictly positive, it immediately follows that $x_1(p_1, p_2, I)$ is strictly positive.

For $x_2(p_1, p_2, I)$ we have

$$\begin{aligned} x_2(p_1, p_2, I) > 0 \iff \frac{I}{p_2} - \frac{p_2}{4p_1} &> 0 \\ \iff 4Ip_1 &> p_2^2 \\ \iff 2\sqrt{Ip_1} &> p_2. \end{aligned}$$

That is, $x_2(p_1, p_2, I)$ is strictly positive if and only if $p_2 < 2\sqrt{Ip_1}$. $\blacksquare$

(e) The condition you derived in (d) suggests that the constraint $x_2 \geq 0$ is a potential problem, that is, it is binding for a subset of the parameters. This is a case in which the utility maximization problem has a corner solution. This type of solutions come about when the parameters of the problem (p_1, p_2, I) are such that the slope of the budget constraint is never equal to that of an indifference curve. Now assume that the condition you derived in (d) does NOT hold and derive the Marshallian demand functions $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$ for this case.

Solution

Now suppose that $p_2 \geq 2\sqrt{Ip_1}$. In this case we have a corner solution, in which all income is spent on a good. The two possible corner solutions are $\left(\frac{I}{p_1}, 0\right)$, in which all income is spent on good 1, and $\left(0, \frac{I}{p_2}\right)$, in which all income is spent on good 2.

Observe that

$$\begin{aligned}
u\left(\frac{I}{p_1}, 0\right) > u\left(0, \frac{I}{p_2}\right) &\iff \sqrt{\frac{I}{p_1}} + 0 > \sqrt{0} + \frac{I}{p_2} \\
&\iff \sqrt{\frac{I}{p_1}} > \frac{I}{p_2} \\
&\iff \frac{I}{p_1} > \frac{I^2}{p_2^2} \\
&\iff p_2^2 > Ip_1 \\
&\iff p_2 > \sqrt{Ip_1}.
\end{aligned}$$

Since the last inequality in the previous chain of equivalences ($p_2 > \sqrt{Ip_1}$) always holds under the assumption $p_2 \geq 2\sqrt{Ip_1}$, we have that in this case $u\left(\frac{I}{p_1}, 0\right) > u\left(0, \frac{I}{p_2}\right)$. Therefore, the Marshallian demand functions are given by

$$x_1(p_1, p_2, I) = \frac{I}{p_1} \qquad \text{and} \qquad x_2(p_1, p_2, I) = 0.$$

for $p_2 \geq 2\sqrt{Ip_1}$. ■

(f) Let's now put things together. Let p_1 and I be fixed. Express and draw the Marshallian demand for good 1 and the Marshallian demand for good 2 as functions of price p_2. (*Hint:* The graph you will obtain should show that the demand for either of the goods, as a function of p_2 when I and p_1 are kept constant, has a point of non-differentiabibility; but despite this, it is continuous.)

Solution

Fix $p_1 > 0$ and $I > 0$. The Marshallian demands for good 1 and good 2 as functions of price p_2 are given by

$$x_1(p_1, p_2, I) = \begin{cases} \left(\frac{p_2}{2p_1}\right)^2 & \text{if } 0 < p_2 < 2\sqrt{Ip_1} \\ \frac{I}{p_1} & \text{if } p_2 \geq 2\sqrt{Ip_1} \end{cases}$$

and

$$x_2(p_1, p_2, I) = \begin{cases} \frac{I}{p_2} - \frac{p_2}{4p_1} & \text{if } 0 < p_2 < 2\sqrt{Ip_1} \\ 0 & \text{if } p_2 \geq 2\sqrt{Ip_1} \end{cases},$$

respectively. ■

Problem 3

Consider once again an economy in which there are two goods, 1 and 2, whose prices are $p_1 > 0$ and $p_2 > 0$, respectively. The two goods can only be consumed in non-negative amounts x_1 and x_2, respectively. A consumer has preferences over $\mathbb{R}^2_+$ which are represented by the utility function

$$u: \mathbb{R}^2_+ \to \mathbb{R}, \quad (x_1, x_2) \mapsto u(x_1, x_2) := \left(\alpha x_1^{-2} + \beta x_2{}^{-2}\right)^{-1/2},$$

where α and β are strictly positive real numbers. The consumer's income is $I > 0$. (Note: This is an example of CES utility function)

(a) Formulate the consumer's utility maximization problem, find the first-order conditions for utility maximization, and find the Marshallian demand functions $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$ for goods 1 and 2, respectively. (Note: Use the Lagrangian method. Assume that the budget constraint holds with equality and that the solution is interior (i.e. $x_1 > 0$ and $x_2 > 0$), thus disregarding the non-negativity constraints on x_1 and x_2. Do not check second order conditions.)

Solution

The agent's utility maximization problem is:

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \left(\alpha x_1^{-2} + \beta x_2^{-2}\right)^{-\frac{1}{2}}$$

$$\text{subject to} \quad p_1 x_1 + p_2 x_2 \leq I.$$

Assuming that the budget constraint holds with equality and disregarding the non-negativity constraints on x_1 and x_2, we can set up the Lagrangian for the utility maximization problem as

$$L(x_1, x_2, \lambda) = \left(\alpha x_1^{-2} + \beta x_2^{-2}\right)^{-\frac{1}{2}} + \lambda\left(p_1 x_1 + p_2 x_2 - I\right),$$

where λ is the Lagrange multiplier.

The first order (necessary) conditions for an interior solution to this problem are:

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0 \iff \left(\alpha x_1^{-3}\right)\left(\alpha x_1^{-2} + \beta x_2^{-2}\right)^{-\frac{3}{2}} + \lambda p_1 = 0$$

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \iff \left(\beta x_2^{-3}\right)\left(\alpha x_1^{-2} + \beta x_2^{-2}\right)^{-\frac{3}{2}} + \lambda p_2 = 0$$

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0 \iff p_1 x_1 + p_2 x_2 - I = 0$$

The optimal choices of x_1 and x_2 solve these three equations simultaneously (also note that there are three unknowns since the value of λ is unknown).

From the first two first order conditions we get

$$\frac{x_1}{x_2} = \left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}}$$

which we can solve for x_1 to obtain

$$x_1 = x_2 \left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}}. \tag{11}$$

Replacing (11) into the first order condition for λ we obtain

$$p_1 x_2 \left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}} + p_2 x_2 = I.$$

Solving the previous expression for x_2 returns the Marshallian demand for good 2:

$$x_2(p_1, p_2, I) = \frac{I}{p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}} + p_2} = \frac{I}{\left(\frac{\beta}{\alpha}\right)^{-\frac{1}{3}} p_1^{\frac{2}{3}} p_2^{\frac{1}{3}} + p_2}. \tag{12}$$

Replacing (12) into (11) gives the Marshallian demand function for good 1:

$$x_1(p_1, p_2, I) = \frac{\left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}} I}{p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{-\frac{1}{3}} + p_2} = \frac{I}{p_1 + p_2 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{3}}} = \frac{I}{p_1 + \left(p_1 \frac{\beta}{\alpha}\right)^{\frac{1}{3}} p_2^{\frac{2}{3}}}. \quad \blacksquare$$

(b) Find the price elasticity of demand for good 1. Is demand for this good elastic or inelastic?

Solution

To find the price elasticity of demand for good 1, we can work with the expression for the total expenditure on good 1, i.e. with

$$p_1 x_1(p_1, p_2, I) = \frac{I}{1 + \left(\frac{\beta}{\alpha}\right)^{\frac{1}{3}} p_2^{\frac{2}{3}} p_1^{-\frac{2}{3}}}. \tag{13}$$

Differentiating this expression with respect to p_1, we get:

$$\begin{aligned}\frac{\partial p_1 x_1(p_1, p_2, I)}{\partial p_1} &= -\frac{I}{D^2} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{3}} p_2^{\frac{2}{3}} \left(-\frac{2}{3}\right) p_1^{-\frac{5}{3}} \\ &= \frac{2I}{3D^2} \left(\frac{\beta}{\alpha}\right)^{\frac{1}{3}} p_2^{\frac{2}{3}} p_1^{-\frac{5}{3}},\end{aligned}$$

where D is the denominator of the expression in (13). The sign of this derivative is positive (by assumption, $p_1, p_2, I, \alpha, \beta > 0$), indicating that demand is inelastic. To see why a positive derivative for total expenditure indicates an elasticity of less than one and therefore inelastic demand, note that

$$\frac{\partial p_1 x_1}{\partial p_1} = x_1 + p_1 \frac{\partial x_1}{\partial p_1},$$

which is equivalent to

$$\frac{\partial p_1 x_1}{\partial p_1} = x_1\left(1 + \frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}\right) = x_1\left(1 - e\right),$$

where $e := -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$ is the (own) price elasticity of demand.

If demand is inelastic ($e < 1$), the derivative of total expenditure is positive, and viceversa. Therefore, in this case we established that demand is inelastic. ■

(c) Find the derivative of the quantity of good 2 demanded with respect to the price of good 1. Use this to indicate whether goods 1 and 2 are substitutes or complements.

Solution

Recall that the Marshallian demand for good 2 is

$$x_2\left(p_1, p_2, I\right) = \frac{I}{\left(\frac{\beta}{\alpha}\right)^{-\frac{1}{3}} p_1^{\frac{2}{3}} p_2^{\frac{1}{3}} + p_2}.$$

The derivative of the quantity of good 2 demanded with respect to the price of good 1 is thus

$$\frac{\partial x_2\left(p_1, p_2, I\right)}{\partial p_1} = -\frac{I}{D^2}\frac{2}{3}\left(\frac{\beta}{\alpha}\right)^{-\frac{1}{3}} p_1^{-\frac{1}{3}} p_2^{\frac{1}{3}} < 0.$$

Therefore the goods are complements.

Now consider a consumer whose preferences over $\mathbb{R}^2_+$ are represented by the CES utility function

$$u\colon \mathbb{R}^2_+ \to \mathbb{R}, \quad (x_1, x_2) \mapsto u(x_1, x_2) := \left(\alpha x_1^\rho + \beta {x_2}^\rho\right)^{1/\rho}, \tag{14}$$

where ρ, α and β are real numbers such that $0 \neq \rho < 1$, $\alpha > 0$ and $\beta > 0$ The consumer's income is $I > 0$. ■

(d) Why does the utility function

$$v\colon \mathbb{R}^2_+ \to \mathbb{R}, \quad (x_1, x_2) \mapsto v(x_1, x_2) := \alpha\frac{x_1^\rho}{\rho} + \beta\frac{x_2^\rho}{\rho}$$

represent the same preferences over $\mathbb{R}^2_+$ as the function u in (14)?

Solution

Utility is an ordinal concept. In class, we argued that utility functions are defined up strictly increasing transformations. This means that if the utility function $u\colon X \to \mathbb{R}$, $x \mapsto u(x)$ represents preferences over a nonempty set X, and the function $f\colon u(X) \to \mathbb{R}$, where $u(X)$ is the range of u, is strictly increasing on $u(X)$, then $v\colon X \to \mathbb{R}$, $v(x) := f(u(x))$ also represents those same preferences over X.[14]

[14] The range $u(X)$ of the function $u\colon X \to \mathbb{R}$, $x \mapsto u(x)$ is defined as $u(X) := \{t \in \mathbb{R} : t = u(x) \text{ for some } x \in X\}$. Observe that $u(X) \subseteq \mathbb{R}$. If the function f is differentiable on $u(X)$, an easy way to check that it is strictly increasing on $u(X)$ is to show that $f'(t) > 0$ for all t in $u(X)$.

Since $\alpha, \beta > 0$, the range $u\left(\mathbb{R}_+^2\right)$ of u is $\mathbb{R}_+$. Define $f: \mathbb{R}_+ \to \mathbb{R}$ as $t \mapsto f(t) := \frac{1}{\rho} t^\rho$ and observe that

$$\begin{aligned} f\left(u(x_1, x_2)\right) &= \frac{1}{\rho}\left(\left(\alpha x_1^\rho + \beta x_2{}^\rho\right)^{1/\rho}\right)^\rho \\ &= \frac{1}{\rho}\left(\alpha x_1^\rho + \beta x_2{}^\rho\right) \\ &= \alpha \frac{x_1^\rho}{\rho} + \beta \frac{x_2^\rho}{\rho} \\ &= v(x_1, x_2), \end{aligned}$$

which shows that v is a transformation of u. To show that v represents the same preferences as u, we only need to show that the transformation f is strictly increasing on $\mathbb{R}_+$. This is easily shown by observing that

$$f'(t) = \frac{1}{\rho} \rho t^{\rho - 1} = t^{\rho - 1} > 0$$

for all real numbers $t > 0$.

Now forget about the previous utility functions. For the next question, consider generic Marshallian demand functions for goods 1 and 2, $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$, resulting from a consumer's utility maximization problem. Assume that Marshallian demands are differentiable functions of prices. ■

(e) Write the budget constraint as

$$p_1 x_1(p_1, p_2, I) + p_2 x_2(p_1, p_2, I) = I.$$

Now take a derivative of this identity with respect to p_1. (Note: Your derivative should have three terms, two of which are related to elasticity and one of which tells you whether these goods are substitutes or complements.) Use this derivative to argue that, for any demand function, if the demand for good 1 is inelastic, then the goods are complements. (Note: This tight relationship does not generalize beyond the case of two goods. This is an unusual case in which a result depends on there being only two goods in our model.)

Solution

Writing the budget constraint using the Marshallian demand function we get

$$p_1 x_1(p_1, p_2, I) + p_2 x_2(p_1, p_2, I) = I.$$

Differentiating the previous identity with respect to p_1 yields

$$x_1 + p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_2}{\partial p_1} = 0$$

Dividing both sides of the previous identity by x_1, we obtain that

$$1 + \frac{p_1}{x_1} \frac{\partial x_1}{\partial p_1} + \frac{p_2}{x_1} \frac{\partial x_2}{\partial p_1} = 0$$

or, equivalently,

$$1 - e + \frac{p_2}{x_1} \frac{\partial x_2}{\partial p_1} = 0,$$

where $e := -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$ is the (own) price elasticity of demand.

Now suppose that the demand for good 1 is inelastic, i.e. $e < 1$. Therefore, $1 - e > 0$, and so the first term on the left hand side is positive. Since the sum of the two terms equals zero and $\frac{p_2}{x_1} > 0$, the term $\frac{\partial x_2}{\partial p_1}$ must be negative, which means the two goods are complements. ■

Problem 4

As you started to see in this course, concave functions and convex sets play an important role in economic analysis. This problem will help you to gain a better grasp of these concepts.

In what follows, whenever convenient for notation, we denote a generic element $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ simply as x. For any $\lambda \in \mathbb{R}$ and any $x \in \mathbb{R}^n$, the scalar multiplication between λ and x, denoted by λx, is the element of $\mathbb{R}^n$ defined as $\lambda x(\lambda x_1, \ldots, \lambda x_n)$. For example, if $\lambda = 2$ and $x = (3, 5) \in \mathbb{R}^2$, then $\lambda x = (6, 10)$.

A subset S of $\mathbb{R}^n$ is said to be convex if for all $x, y \in S$ and all $\lambda \in [0, 1]$, we have $\lambda x + (1-\lambda)y \in S$. In words, the definition says that a subset S of $\mathbb{R}^n$ is convex if for any two elements x and y belonging to S there are no elements of the straight line segment between x and y that are not elements of S. An expression of the form $\lambda x + (1 - \lambda)y$, where $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, is called a convex combination of x and y.

(a) Suppose that A and B are arbitrary convex subsets of $\mathbb{R}^n$. Show that $A \cap B$ is convex, while $A \cup B$ need not be. (*Hint:* You may want to use a graphical representation of the two set operations to guide your reasoning.)

Solution

Let's start by showing that $A \cap B$ is a convex set. Fix two arbitrary elements x and y of $A \cap B$ and an arbitrary λ in $[0, 1]$. To show that $A \cap B$ is convex, we need to show that $\lambda x + (1 - \lambda)y$ is an element of $A \cap B$.

Since x and y are elements of $A \cap B$, by definition of intersection we know that x and y are elements of A. Since A is convex, $\lambda x + (1 - \lambda)y$ is an element of A.

Analogously, since x and y are elements of $A \cap B$, they are also elements of B. Since B is convex, $\lambda x + (1 - \lambda)y$ is an element of B.

Therefore, $\lambda x + (1-\lambda)y$ is both an element of A and an element of B. Therefore, we conclude by definition of intersection that $\lambda x + (1 - \lambda)y$ is an element of $A \cap B$, as we wanted to show.

Now we show that the statement "$\langle$ A convex and B convex $\rangle \Rightarrow A \cup B$ convex" is false.

In general, when you have to show that a statement is false, you just need to provide a counterexample (i.e., an exception to the proposed statement). Sometimes counterexamples are easy to find, but sometimes they are really hard. In our case, finding a counterexample is trivial.

Let x and y be two real numbers such that $x \neq y$. The sets $\{x\}$ and $\{y\}$ are obviously convex subsets of $\mathbb{R}$. However, their union $\{x\} \cup \{y\} = \{x, y\}$ is not. Indeed, for any $\lambda \in (0, 1)$ we have $\lambda x + (1 - \lambda)y \notin \{x, y\}$.

For $n \geq 2$, we observe that $\{x\} \times \mathbb{R}^{n-1}$ and $\{y\} \times \mathbb{R}^{n-1}$ are convex subsets of $\mathbb{R}^n$, while their union $\left(\{x\} \times \mathbb{R}^{n-1}\right) \cup \left(\{y\} \times \mathbb{R}^{n-1}\right) = \{x, y\} \times \mathbb{R}^{n-1}$ is not for an analogous argument as the one given above. ■

(b) Consider an economy in which there are two goods, 1 and 2, whose prices are $p_1 > 0$ and $p_2 > 0$, respectively. The two goods can only be consumed in non-negative amounts x_1 and x_2. The budget set of a consumer with income $I > 0$ is defined as

$$B(p_1, p_2, I) := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq I\right\}.$$

Show that $B(p_1, p_2, I)$ is a convex subset of $\mathbb{R}^2_+$.

Solution

Fix two arbitrary elements $x := (x_1, x_2)$ and $y := (y_1, y_2)$ of $B(p_1, p_2, I)$ and an arbitrary λ in $[0, 1]$. To show that $B(p_1, p_2, I)$ is convex, we need to show that $\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2)$ is an element of $B(p_1, p_2, I)$. Given the definition of $B(p_1, p_2, I)$, this amounts to prove that the following three inequalities are satisfied

$$\lambda x_1 + (1-\lambda)y_1 \geq 0,$$

$$\lambda x_2 + (1-\lambda)y_2 \geq 0,$$

$$p_1[\lambda x_1 + (1-\lambda)y_1] + p_2[\lambda x_2 + (1-\lambda)y_2] \leq I.$$

For the first inequality, observe that since (x_1, x_2) and (y_1, y_2) are elements of $B(p_1, p_2, I)$, $x_1 \geq 0$ and $y_1 \geq 0$. Since $\lambda \in [0, 1]$, so that also $(1-\lambda) \in [0, 1]$, it immediately follows that $\lambda x_1 + (1-\lambda)y_1 \geq 0$. The second inequality is established analogously. For the third inequality, observe that

$$\begin{aligned} p_1[\lambda x_1 + (1-\lambda)y_1] + p_2[\lambda x_2 + (1-\lambda)y_2] &= p_1\lambda x_1 + p_1(1-\lambda)y_1 + p_2\lambda x_2 + p_2(1-\lambda)y_2 \\ &= \lambda(p_1 x_1 + p_2 x_2) + (1-\lambda)(p_1 y_1 + p_2 y_2) \\ &\leq \lambda I + (1-\lambda)I \\ &= I, \end{aligned}$$

where the inequality holds because (x_1, x_2) and (y_1, y_2) are elements of $B(p_1, p_2, I)$, and so $p_1 x_1 + p_2 x_2 \leq I$ and $p_1 y_1 + p_2 y_2 \leq I$. This completes the proof.

Let S be a convex subset of $\mathbb{R}^n$. A function $f : S \to \mathbb{R}$, $x \mapsto f(x)$ is said to be concave if for all $x, y \in S$ and all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

Let S be a convex subset of $\mathbb{R}^n$. A function $f : S \to \mathbb{R}$, $x \mapsto f(x)$ is said to be quasi-concave if for all $x, y \in S$ and all $\lambda \in [0, 1]$, we have[15]

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}. \qquad \blacksquare$$

(c) Give a graphical example of a concave function $f : S \to \mathbb{R}$, $x \mapsto f(x)$, where S is a convex subset of $\mathbb{R}$. Clearly illustrate the role of the convex combination $\lambda x + (1-\lambda)y$ of x and y for all $\lambda \in [0, 1]$. Is this function also quasi-concave?

(d) Now draw a function $f : S \to \mathbb{R}$, $x \mapsto f(x)$, where $S \subseteq \mathbb{R}$ and convex, which is quasi-concave, but fails to be concave. What is the difference between concave and quasi-concave functions?

[15] *Quiz (OPTIONAL):* Do you see why, when defining concave or quasi-concave functions, we require their domain (i.e., the set S in the current definition) to be a convex set?

(e) Argue formally that every concave function is also quasi-concave. (Note: The example you provided in (d) shows that the converse statement is false.)

Solution

Let S be a convex subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}$, $x \mapsto f(x)$ be a concave function on S. Fix two arbitrary elements x and y of S and an arbitrary $\lambda \in [0, 1]$. Then, we have

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &\geq \lambda f(x) + (1-\lambda) f(y) \\ &\geq \lambda \min\{f(x), f(y)\} + (1-\lambda) \min\{f(x), f(y)\} \\ &= \min\{f(x), f(y)\}, \end{aligned}$$

where the first inequality holds because f is a concave function, and the second inequality holds because $f(x) \geq \min\{f(x), f(y)\}$, $f(y) \geq \min\{f(x), f(y)\}$ and $\lambda, 1-\lambda \geq 0$. Therefore, f is quasi-concave. ■

ECON 121: Intermediate Microeconomics
Solutions to Problem Set 6

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Problem 1

Consider an exchange economy in which there are only two agents, Robinson and Friday, and two goods, 1 and 2. Denote with $\left(w_1^R, w_2^R\right) \in \mathbb{R}_+^2$ and $\left(w_1^F, w_2^F\right) \in \mathbb{R}_+^2$ the initial endowments of Robinson and Friday, respectively. The preferences of the two agents are described by the real-valued utility functions

$$u^R\left(x_1^R, x_2^R\right) := 2\sqrt{x_1^R} + x_2^R,$$

and

$$u^F\left(x_1^F, x_2^F\right) := x_1^F + 2\sqrt{x_2^F},$$

defined on $\mathbb{R}_+^2$.

Q.1 Compute the demand functions of Robinson and Friday. [*Note:* Assume that the budget constraint holds with equality, that the solution is interior and that the second order conditions are satisfied. Moreover, assume that prices are strictly positive.]

Solution

Robinson's utility maximization problem is

$$\max_{\left(x_1^R, x_2^R\right)\in\mathbb{R}_+^2} 2\sqrt{x_1^R} + x_2^R$$

$$\text{s.t.} \quad p_1 x_1^R + p_2 x_2^R = p_1 w_1^R + p_2 w_2^R.$$

The Lagrangian for this constrained maximization problem is

$$L\left(x_1^R, x_2^R, \lambda\right) = 2\sqrt{x_1^R} + x_2^R + \lambda\left[p_1 x_1^R + p_2 x_2^R - p_1 w_1^R - p_2 w_2^R\right],$$

where λ is the Lagrange multiplier. The first order (necessary) conditions for an interior solution to the problem are

$$\frac{\partial L}{\partial x_1^R} = 0 \Longleftrightarrow \frac{1}{\sqrt{x_1^R}} + \lambda p_1 = 0, \tag{1}$$

$$\frac{\partial L}{\partial x_2^R} = 0 \Longleftrightarrow 1 + \lambda p_2 = 0, \tag{2}$$

and
$$\frac{\partial L}{\partial \lambda} = 0 \Longleftrightarrow p_1 x_1^R + p_2 x_2^R - p_1 w_1^R - p_2 w_2^R = 0. \tag{3}$$

Equation (2) implies that $\lambda = -\frac{1}{p_2}$ (provided $p_2 > 0$). Plugging this into Equation (1) gives us the the demand function for good 1:

$$x_1^R(p_1, p_2, w_1^R, w_2^R) = \left(\frac{p_2}{p_1}\right)^2.$$

Replacing $x_1^R(p_1, p_2, w_1^R, w_2^R)$ for x_1^R in Equation (3), we obtain the demand function for good 2:

$$x_2^R(p_1, p_2, w_1^R, w_2^R) = \frac{w_1^R p_1}{p_2} + w_2^R - \frac{p_2}{p_1}.$$

Friday's utility maximization problem is

$$\max_{(x_1^F, x_2^F) \in \mathbb{R}_+^2} x_1^F + 2\sqrt{x_2^F}$$
$$\text{s.t.} \quad p_1 x_1^F + p_2 x_2^F = p_1 w_1^F + p_2 w_2^F.$$

The Lagrangian associated to this constrained maximization problem is

$$L(x_1^F, x_2^F, \lambda) = x_1^F + 2\sqrt{x_2^F} + \lambda[p_1 x_1^F + p_2 x_2^F - p_1 w_1^F - p_2 w_2^F],$$

where λ is the Lagrange multiplier. The first order (necessary) conditions for an interior solution to the problem are:

$$\frac{\partial L}{\partial x_1^F} = 0 \Longleftrightarrow 1 + \lambda p_1 = 0, \tag{4}$$

$$\frac{\partial L}{\partial x_2^F} = 0 \Longleftrightarrow \frac{1}{\sqrt{x_2^F}} + \lambda p_2 = 0, \tag{5}$$

and
$$\frac{\partial L}{\partial \lambda} = 0 \Longleftrightarrow p_1 x_1^F + p_2 x_2^F - p_1 w_1^F - p_2 w_2^F = 0. \tag{6}$$

Equation (4) implies that $\lambda = -\frac{1}{p_1}$ (provided $p_1 > 0$). Plugging this into Equation (5) gives us the demand function for good 2:

$$x_2^F(p_1, p_2, w_1^F, w_2^F) = \left(\frac{p_1}{p_2}\right)^2.$$

Replacing $x_1^F(p_1, p_2, w_1^F, w_2^F)$ for x_1^F in Equation (6), we obtain the demand function for good 1:

$$x_1^F(p_1, p_2, w_1^F, w_2^F) = \frac{w_2^F p_2}{p_1} + w_1^F - \frac{p_1}{p_2}. \qquad \blacksquare$$

Q.2 Assume that Robinson's endowment is $\left(w_1^R, w_2^R\right) = (2, 0)$, and Friday's endowment is $\left(w_1^F, w_2^F\right) = (0, 2)$. Compute the competitive equilibrium price and allocation. To do so, formulate the market-clearing conditions. These may look complicated. To proceed, remember that: (a) by Walras' law, it is enough to restrict attention to one market clearing condition: (b) you can choose one good (either one) to be the numeraire and set its price equal to 1.

Solution

The market clearing condition for good 1 requires the aggregate demand for good 1 to equal the aggregate endowment of good 1. That is,

$$x_1^R + x_1^F = w_1^R + w_1^F.$$

Similarly, for good 2

$$x_2^R + x_2^F = w_2^R + w_2^F.$$

By Walras' law, in an economy with n goods, it suffices to consider only $n-1$ market clearing conditions (as the condition for the n-th market would be redundant). Consider good 1. Given the demand functions derived above, the market clearing condition for good 1 in this economy is

$$\left(\frac{p_2}{p_1}\right)^2 + \frac{w_2^F p_2}{p_1} + w_1^F - \frac{p_1}{p_2} = w_1^R + w_1^F.$$

Given endowments $\left(w_1^R, w_2^R\right) = (2, 0)$ and $\left(w_1^F, w_2^F\right) = (0, 2)$, the market clearing condition becomes

$$\left(\frac{p_2}{p_1}\right)^2 + \frac{2p_2}{p_1} - \frac{p_1}{p_2} = 2.$$

Since we are always free to choose the price of one good (the numeraire), let's set $p_1 = 1$. The previous equation thus becomes

$$p_2^3 + 2p_2^2 - 2p_2 = 1,$$

which is satisfied for $p_2 = 1$. We conclude that a competitive equilibrium price vector is $(p_1, p_2) = (1, 1)$. Notice that $p_1 > 0$ and $p_2 > 0$, as we assumed.[1]

Substituting the values for prices and endowments in the demand functions, we obtain the competitive equilibrium allocation:

$$\left(x_1^R, x_2^R\right) = (1, 1)$$
$$\left(x_1^F, x_2^F\right) = (1, 1). \qquad \blacksquare$$

Q.3 Compare the utility of each Robinson and Friday at the competitive equilibrium to that which they would receive if they did not trade with one another. Comment your finding (one line).

Solution

Notice that

$$U^R(1, 1) = 3 > 2\sqrt{2} = U^R(2, 0).$$

[1] Any price vector (p_1, p_2) such that $p_1 = p_2$ is a competitive equilibrium price vector for this economy.

Similarly,

$$U^F(1,1) = 3 > 2\sqrt{2} = U^F(0,2).$$

Hence, both Robinson and Friday achieve a higher utility than that they would attain if they did not trade and instead consumed their endowments. All gains from trade exhausted. ■

Q.4 Let us now consider an application. Suppose that we think of Robinson and Friday as two countries. Let their utility functions be the same as before, but let their initial endowments be $\left(w_1^R, w_2^R\right) = (3,9)$ and $\left(w_1^F, w_2^F\right) = (1,3)$. Hence, we can think of Robinson as a relatively rich country and Friday as a relatively poor country, in the sense that Robinson has more of both goods. Find the competitive equilibrium. [*Note:* You don't have to start over from scratch to do this. You can simply repeat your derivations from parts Q.1 and Q.2, once again making the guess that $p_1 = p_2$ and inserting the new values for the endowments.] If the rich country has more of everything, why are there gains from trade? One sometimes hears calls for the U.S. to curtail its foreign trade. Thinking of the U.S. as Robinson, what effect would this have in this model? Given your answer, why would people suggest such curtailment or, equivalently, what would such advocates think of as missing from our model?

Solution

Robinson's and Friday's endowments are now $\left(w_1^R, w_2^R\right) = (3,9)$ and $\left(w_1^F, w_2^F\right) = (1,3)$. Substituting the new values for the endowments in the market clearing condition for good 1, we obtain

$$\left(\frac{p_2}{p_1}\right)^2 + \frac{3p_2}{p_1} - \frac{p_1}{p_2} = 3,$$

which is again satisfied at any price vector (p_1, p_2) such that $p_1 = p_2$. Therefore, any price vector satisfying $p_1 = p_2$ is a competitive equilibrium price vector.

Plugging the values for prices and the new endowments in the demand functions, we obtain the competitive equilibrium allocation:

$$\left(x_1^R, x_2^R\right) = (1, 11),$$

and

$$\left(x_1^F, x_2^F\right) = (3, 1).$$

Notice that

$$U^R(1,11) = 13 > 2\sqrt{3} + 9 = U^R(3,9),$$

and

$$U^F(3,1) = 5 > 1 + 2\sqrt{3} = U^F(1,3).$$

That is, both the rich country and the poor country are better off with trade (gains from trade). Trade makes the rich country better off, even though such country already has more of both goods, because it allows the country to obtain the optimal mix of goods (in this case, consuming more of good 2 than the endowment would allow).

If we think of the U.S. as the rich country in this model, then a policy that curtailed foreign trade would make the U.S. worse off. Note that our framework models an exchange economy and abstracts away from production. Advocates of such curtailment policy likely have in mind a framework where the U.S. and poor countries produce and trade goods, which involves additional implications not captured by our model. ■

Problem 2

Solve the following questions.

Q.1 Consider the following Edgeworth box economy. There are two consumers, A, and B, and two goods, 1 and 2. The two consumers have identical preferences represented by the real-valued utility functions

$$u^j\left(x_1^j, x_2^j\right) := \min\left\{x_1^j, x_2^j\right\},$$

$j = A, B$, defined on $\mathbb{R}^2_+$. There are 20 units of good 1 and 10 units of good 2. Characterize the set of Pareto efficient allocations.

Solution

Pareto efficient allocations $\left(\left(x_1^A, x_2^A\right), \left(x_1^B, x_2^B\right)\right)$ are characterized by the following conditions:

$$x_1^A \geq x_2^A, \qquad x_1^B \geq x_2^B, \qquad x_1^A + x_1^B \leq 20, \qquad x_2^A + x_2^B = 10. \qquad ■$$

Q.2 Consider an exchange economy with two consumers, A and B, and two goods, 1 and 2. The two consumers have identical preferences represented by the utility functions

$$u^j\left(x_1^j, x_2^j\right) := x_1^j x_2^j$$

$j = A, B$, defined on $\mathbb{R}^2_+$. for $j = A, B$. Their initial endowments are $\left(w_1^A, w_2^A\right) = (1, 1)$ and $\left(w_1^B, w_2^B\right) = (1, 3)$. Compute the competitive equilibrium price and allocation.

Solution

As only relative prices matter, we normalize $p_2 = 1$. Consumer A's utility maximization problem is

$$\max_{\left(x_1^A, x_2^A\right) \in \mathbb{R}^2_+} \; x_1^A x_2^A$$

$$\text{s.t.} \quad p_1 x_1^A + x_2^A = p_1 + 1.$$

Hence, consumer A's demands are

$$x_1^A = \frac{p_1 + 1}{2p_1},$$

and

$$x_2^A = \frac{p_1 + 1}{2}.$$

Similarly, we find consumer B's demands:

$$x_1^B = \frac{p_1 + 3}{2p_1},$$

and

$$x_2^B = \frac{p_1 + 3}{2}.$$

Market clearing conditions require that

$$x_1^A + x_1^B = w_1^A = w_1^B.$$

Given A's and B's demand functions and initial endowments for good 1, the market clearing condition for good 1 becomes

$$\frac{p_1 + 1}{2p_1} + \frac{p_1 + 3}{2p_1} = 2,$$

i.e.

$$\frac{2p_1 + 4}{2p_1} = 2,$$

i.e. $p_1 = 2$.

We conclude that any competitive equilibrium price vector (p_1, p_2) satisfies $\frac{p_1}{p_2} = 2$, and that the competitive equilibrium allocation is

$$((x_1^A, x_2^A), (x_1^B, x_2^B)) = \left(\left(\frac{3}{4}, \frac{3}{2}\right), \left(\frac{5}{4}, \frac{5}{2}\right)\right). \qquad \blacksquare$$

Q.3 An exchange economy has three consumers (A, B and C), and three goods (1, 2 and 3). Consumers' utility functions and initial endowments are as follows:

$$u^A(x_1^A, x_2^A, x_3^A) := \min\{x_1^A, x_2^A\}, \qquad (w_1^A, w_2^A, w_3^A) = (1, 0, 0);$$

$$u^B(x_1^B, x_2^B, x_3^B) := \min\{x_2^B, x_3^B\}, \qquad (w_1^B, w_2^B, w_3^B) = (0, 1, 0);$$

$$u^C(x_1^C, x_2^C, x_3^C) := \min\{x_1^C, x_3^C\}, \qquad (w_1^C, w_2^C, w_3^C) = (0, 0, 1).$$

Find the competitive equilibrium price and allocation.

Solution

Since only relative prices matter, let's normalize $p_1 = 1$. Consumer A's demand satisfies

$$x_1^A = x_2^A \qquad \text{and} \qquad x_1^A + p_2 x_2^A = 1.$$

Hence,

$$x_1^A = x_2^A = \frac{1}{1 + p_2}.$$

Similarly, for consumer B and consumer C we obtain

$$\left\langle x_2^B = x_3^B \quad \text{and} \quad p_2 x_2^B + p_3 x_3^B = p_2 \right\rangle \implies x_2^B = x_3^B = \frac{p_2}{p_2 + p_3},$$

and

$$\left\langle x_1^C = x_3^C \quad \text{and} \quad x_1^C + p_3 x_3^C = p_3 \right\rangle \implies x_1^C = x_3^C = \frac{p_3}{1 + p_3}.$$

Together with the market clearing conditions, we have

$$\begin{cases} \frac{1}{1+p_2} + \frac{p_3}{1+p_3} = 1 \\ \frac{1}{1+p_2} + \frac{p_2}{p_2+p_3} = 1 \end{cases},$$

which implies that $p_2 = p_3 = 1$.

Hence, a competitive equilibrium price vector is $(1, 1, 1)$, and the competitive equilibrium allocation is

$$\left(\left(x_1^A, x_2^A, x_3^A\right), \left(x_1^B, x_2^B, x_3^B\right), \left(x_1^C, x_2^C, x_3^C\right)\right) = \left(\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right)\right). \qquad \blacksquare$$

Problem 3

Consider an exchange economy in which there are two agents, A and B, and two goods, 1 and 2. The two agents have preferences described by the real-valued utility functions

$$u^A\left(x_1^A, x_2^A\right) := \alpha \ln x_1^A + (1-\alpha) \ln x_2^A,$$

and

$$u^B\left(x_1^B, x_2^B\right) := (1-\alpha) \ln x_1^B + \alpha \ln x_2^B,$$

where $\alpha \in (1/2, 1)$. Suppose that consumers start with initial endowments $\left(w_1^A, w_2^A\right) = \left(w_1^B, w_2^B\right) = (1, 1)$. The agents have agreed to share their resources. They have also agreed that the weight that A receives in the economy is $\gamma_A \in (0, 1)$, and the weight that B receives is $\gamma_B := 1 - \gamma_A$. [*Note:* For this problem, feel free to be loose and ignore the fact that the natural logarithm function is not defined at zero.]

Q.1 For every weight $\gamma_A \in (0, 1)$, find the allocation which would maximize the social surplus given the weights; that is, we are interested in finding the allocation $\left(\left(x_1^A, x_2^A\right), \left(x_1^B, x_2^B\right)\right)$ which maximizes the sum

$$\gamma_A u^A\left(x_1^A, x_2^A\right) + \left(1-\gamma_A\right) u^B\left(x_1^B, x_2^B\right)$$

subject to the resource constraints of the economy.

Solution

The relevant constrained maximization problem is given by

$$\max_{\left(x_1^A, x_2^A, x_1^B, x_2^B\right) \in \mathbb{R}_+^4} \gamma_A\left(\alpha \ln x_1^A + (1-\alpha) \ln x_2^A\right) + \left(1-\gamma_A\right)\left((1-\alpha) \ln x_1^B + \alpha \ln x_2^B\right)$$

$$\text{s.t. } x_1^A + x_1^B = 2$$
$$x_2^A + x_2^B = 2.$$

The first order conditions (after eliminating the Lagrange multipliers) are

$$\begin{aligned} \gamma_A \alpha x_1^B &= \left(1-\gamma_A\right)(1-\alpha) x_1^A, \\ \gamma_A (1-\alpha) x_2^B &= \left(1-\gamma_A\right) \alpha x_2^A, \end{aligned}$$

$$\begin{aligned} x_1^A + x_1^B &= 2, \\ x_2^A + x_2^B &= 2. \end{aligned}$$

Solve this system of equations to obtain

$$x_1^A = \frac{2\gamma_A\alpha}{\gamma_A\alpha + (1-\gamma_A)(1-\alpha)}, \tag{7}$$

$$x_2^A = \frac{2\gamma_A(1-\alpha)}{\gamma_A(1-\alpha) + (1-\gamma_A)\alpha}, \tag{8}$$

$$x_1^B = \frac{2(1-\gamma_A)(1-\alpha)}{\gamma_A\alpha + (1-\gamma_A)(1-\alpha)}, \tag{9}$$

$$x_2^B = \frac{2(1-\gamma_A)\alpha}{\gamma_A(1-\alpha) + (1-\gamma_A)\alpha}. \quad \blacksquare \tag{10}$$

Q.2 For every weight $\gamma_A \in (0,1)$, find initial endowments of goods 1 and 2 among agents A and B and a pair of prices such that the Pareto efficient allocation actually constitutes an equilibrium of the market. [Here we decentralize the Pareto efficient allocation via a market equilibrium.]

Solution

For every weight $\gamma_A \in (0,1)$, let the initial endowment be given by the system of Pareto efficient allocations in (7)-(10). Normalize the price of good 1 to $p_1 = 1$. We need to find the price of good 2 that together with endowments given by (7)-(10) will constitute a competitive equilibrium.

Since (7)-(10) is a Pareto efficient allocation, the marginal rate of substitution between good 1 and good 2 are the same for agent A and agent B (verify by yourself that this is actually true). Next, recall that, at a competitive equilibrium, the marginal rate of substitution equals the price ratio. We will use this property to recover the price for good 2:

$$\begin{aligned} MRS^B_{2,1}(x_1^B, x_2^B) &= \frac{p_2}{p_1} \\ \Longleftrightarrow \quad \frac{\alpha x_1^B}{(1-\alpha)x_2^B} &= p_2 \\ \Longleftrightarrow \quad \frac{\alpha \frac{2(1-\gamma_A)(1-\alpha)}{\gamma_A\alpha+(1-\gamma_A)(1-\alpha)}}{(1-\alpha)\frac{2(1-\gamma_A)\alpha}{\gamma_A(1-\alpha)+(1-\gamma_A)\alpha}} &= p_2 \\ \Longleftrightarrow \quad \frac{\gamma_A(1-\alpha)+(1-\gamma_A)\alpha}{\gamma_A\alpha+(1-\gamma_A)(1-\alpha)} &= p_2. \end{aligned}$$

Thus, the price vector

$$(p_1, p_2) = \left(1, \frac{\gamma_A(1-\alpha)+(1-\gamma_A)\alpha}{\gamma_A\alpha+(1-\gamma_A)(1-\alpha)}\right)$$

and the allocation given by (7)-(10) constitute a competitive equilibrium in an economy where initial endowments are given by (7)-(10). It will be just a simple exchange economy where everybody consumes his own endowment at the equilibrium prices. $\blacksquare$